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A summation formula over the zeros of a combination of the associated Legendre functions with a physical application

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Abstract
By using the generalized Abel–Plana formula, we derive a summation formula for the series over the zeros of a combination of the associated Legendre functions with respect to the degree. The summation formula for the series over the zeros of the combination of the Bessel functions, previously discussed in the literature, is obtained as a limiting case. As an application we evaluate the Wightman function for a scalar field with a general curvature coupling parameter in the region between concentric spherical shells on a background of constant negative curvature space. For the Dirichlet boundary conditions the corresponding mode-sum contains the series over the zeros of the combination of the associated Legendre functions. The application of the summation formula allows us to present the Wightman function in the form of the sum of two integrals. The first one corresponds to the Wightman function for the geometry of a single spherical shell and the second one is induced by the presence of the second shell. The boundary-induced part in the vacuum expectation value of the field squared is investigated. For points away from the boundaries the corresponding renormalization procedure is reduced to that for the boundary-free part.

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1. Introduction

The associated Legendre functions are an important class of special functions that appear in a wide range of problems of mathematical physics. The physical importance of these functions is related to the fact that they appear as solutions of the field theory equations in various situations. In particular, the radial parts of the solutions for the scalar, fermionic and electromagnetic wave equations on background of constant curvature spacetimes are expressed in terms of the associated Legendre functions (see, for instance, [1–3]). The eigenfunctions in braneworld models with de Sitter and anti-de Sitter branes are also expressed in terms of these functions (see [4]). Motivated by this, in [5], by making use of the generalized Abel–Plana formula, we have derived a summation formula for the series over the zeros of the associated Legendre function of the first kind with respect to the degree (for the generalized Abel–Plana formula and its applications to physical problems see [6–8]). This type of series is contained in the mode-sum for two-point functions of a quantum scalar field in the background of a constant curvature space with spherical boundary, on which the field obeys the Dirichlet boundary condition. The application of the summation formula allowed us to extract from the vacuum expectation values the part corresponding to the situation without boundary and to present the boundary-induced part in terms of rapidly convergent integral.

In the corresponding problem with two concentric spherical boundaries, in the region between two spheres the radial part of the eigenfunctions is expressed in terms of a combination of the associated Legendre functions of the first and second kinds. The eigenfrequencies are determined by the location of the zeros of this combination with respect to the degree. In the present paper, by specifying the functions in the generalized Abel–Plana formula, we obtain a summation formula for the series over these zeros. As in the case of the other Abel–Plana-type formulas, previously considered in the literature, this formula presents the sum of the series over the zeros of the combination of the associated Legendre function in the form of the sum of two integrals. In boundary-value problems with two boundaries the first integral corresponds to the situation when one of the boundaries is absent and the second one presents the part induced by the second boundary. For a large class of functions the latter is rapidly convergent and, in particular, is useful for the numerical evaluations of the corresponding physical characteristics.

The paper is organized as follows. In section 2, by specifying the functions in the generalized Abel–Plana formula we derive a formula for the summation of the series over zeros of the combination of the associated Legendre functions with respect to the degree. In section 3, special cases of this summation formula are considered. First, as a partial check we show that as a special case the standard Abel–Plana formula is obtained. Then we show that from the summation formula discussed in section 2, as a limiting case the formula is obtained for the summation of the series over the zeros of the combinations of the Bessel functions, previously derived in [6]. A physical application is given in section 4, where the positive frequency Wightman function for a scalar field is evaluated in the region between two spherical boundaries on the background of a negative constant curvature space. It is assumed that the field obeys the Dirichlet boundary condition on the spherical shells. The use of the summation formula from section 2 allows us to extract from the vacuum expectation value the part corresponding to the geometry where the outer sphere is absent. The part induced by the latter is presented in terms of an integral, which is rapidly convergent in the coincidence limit for points away from the sphere. The main results of the paper are summarized in section 5. In the appendix the formula for the normalization integral is derived and we show that the zeros of the combination of the associated Legendre functions with respect to the degree are simple.

2. Summation formula

Let $z = z_k$, $k = 1, 2, \ldots$, be zeros of the function

$$X^{\mu}_{\mathrm{i}z}(u, v) = \frac{P^{\mu}_{\mathrm{i}z-1/2}(u)P^{-\mu}_{\mathrm{i}z-1/2}(v) - P^{-\mu}_{\mathrm{i}z-1/2}(u)P^{\mu}_{\mathrm{i}z-1/2}(v)}{\sin(\mu\pi)}, \tag{1}$$

in the right half-plane of the complex variable z;

$$X^{\mu}_{\mathrm{i}z_k}(u, v) = 0. \tag{2}$$

In (1), $P^{\mu}_{\mathrm{i}z-1/2}(u)$ is the associated Legendre function of the first kind (in this paper the definition of the associated Legendre functions follows that given in [9]). In the discussion below we will assume that $u, v > 1$. The expression in the numerator of (1) has simple zeros for integer values of μ and the function $X^{\mu}_{\mathrm{i}z}(u, v)$ is regular at these points. Since one has the property $X^{-\mu}_{\nu}(u, v) = X^{\mu}_{\nu}(u, v)$, without loss of generality, we consider the parameter μ being non-negative, $\mu \geqslant 0$. For given values u, v, and μ the function $X^{\mu}_{iz}(u, v)$ has an infinity of real zeros. From the asymptotic formula for the associated Legendre functions, we can see that for $z \to +\infty$ one has

$$X^{\mu}_{\mathrm{i}z}(u, v) \approx \frac{2\sin[(\eta_v - \eta_u)\, z]}{\pi z\sqrt{\sinh\eta_u \sinh\eta_v}}, \tag{3}$$

where η_u and η_v are defined as

$$u = \cosh\eta_u, \qquad v = \cosh\eta_v. \tag{4}$$

From here we obtain the asymptotic expression for large zeros:

$$z_k \approx \pi k/(\eta_v - \eta_u). \tag{5}$$

In general, the zeros z_k are functions of the parameters u, v, and μ: $z_k = z_k(u, v, \mu)$. By taking into account that for the associated Legendre function one has $P^{\mu}_{-\nu-1/2}(u) = P^{\mu}_{\nu-1/2}(u)$, we see that $X^{\mu}_{-\nu}(u, v) = X^{\mu}_{\nu}(u, v)$. Hence, the points $z = -z_k$ are zeros of the function $X^{\mu}_{\mathrm{i}z}(u, v)$ as well. In the appendix we show that the zeros $z = z_k$ are simple and under the conditions specified above the function $X^{\mu}_{\mathrm{i}z}(u, v)$ has no zeros which are not real. We will assume that z_k are arranged in ascending order of magnitude. Note that the function $X^{\mu}_{\nu}(u, v)$ can also be expressed in terms of the combination

$$Y^{\mu}_{\nu}(u, v) = Q^{\mu}_{\nu-1/2}(u)P^{\mu}_{\nu-1/2}(v) - P^{\mu}_{\nu-1/2}(u)Q^{\mu}_{\nu-1/2}(v), \tag{6}$$

as

$$X^{\mu}_{\nu}(u, v) = \frac{2}{\pi \mathrm{e}^{\mathrm{i}\mu\pi}} \frac{\Gamma(\nu - \mu + 1/2)}{\Gamma(\nu + \mu + 1/2)} Y^{\mu}_{\nu}(u, v), \tag{7}$$

where $Q^{\mu}_{\nu-1/2}(u)$ is the associated Legendre function of the second kind and $\Gamma(x)$ is the gamma function.

A summation formula for the series over z_k can be derived by using the generalized Abel–Plana formula [6] (see also [7, 8]). For functions $f(z)$ and $g(z)$ meromorphic in the strip $a \leqslant x \leqslant b$ of the complex plane $z = x + \mathrm{i}y$ this formula has the form

$$\lim_{b\to\infty}\left[\text{p.v.}\int_a^b \mathrm{d}x\, f(x) - \pi \mathrm{i}\sum_k \operatorname*{Res}_{z=z_{g,k}} g(z) - \pi\,\mathrm{i} \sum_{k,\mathrm{Im}\, z_{f,k}\neq 0} \sigma(z_{f,k}) \operatorname*{Res}_{z=z_{f,k}} f(z)\right]$$
$$= \frac{1}{2}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{d}z\, [g(z) + \sigma(z) f(z)], \tag{8}$$

where $\sigma(z) \equiv \mathrm{sgn}(\mathrm{Im}\, z)$ and p.v. means the principal value of the integral. In this formula, $z_{f,k}$ and $z_{g,k}$ are the positions of the poles of the functions $f(z)$ and $g(z)$ in the strip $a < x < b$. As functions $f(z)$ and $g(z)$ in formula (8) we choose

$$\begin{aligned} f(z) &= \frac{h(z)}{4Q^{\mu}_{\mathrm{i}z-1/2}(u)Q^{\mu}_{-\mathrm{i}z-1/2}(u)} \frac{\Gamma\,(\mathrm{i}z + \mu + 1/2)\, \pi^2 \mathrm{e}^{2\mathrm{i}\mu\pi}\, \mathrm{i}\sinh(z\pi)}{\Gamma\,(\mathrm{i}z - \mu + 1/2)\cos[(\mathrm{i}z - \mu)\pi]}, \\ g(z) &= \left[\frac{Q^{\mu}_{-\mathrm{i}z-1/2}(v)}{Q^{\mu}_{-\mathrm{i}z-1/2}(u)} + \frac{Q^{\mu}_{\mathrm{i}z-1/2}(v)}{Q^{\mu}_{\mathrm{i}z-1/2}(u)}\right] \frac{h(z)}{2X^{\mu}_{\mathrm{i}z}(u, v)}, \end{aligned} \tag{9}$$

where $h(z)$ is a meromorphic function for $a \leqslant \mathrm{Re}\, z \leqslant b$. The combinations appearing on the left-hand side of formula (8) are presented in the form

$$g(z) \pm f(z) = \frac{Q^{\mu}_{\mp iz-1/2}(v)}{Q^{\mu}_{\mp iz-1/2}(u)} \frac{h(z)}{X^{\mu}_{iz}(u,v)}. \tag{10}$$

Note that the function $g(z)$ has simple poles at the zeros z_k of the function (1). With the help of the asymptotic formulas for the associated Legendre functions, we can see that the conditions for the generalized Abel–Plana formula (8) are satisfied if the function $h(z)$ is restricted by the constraint

$$|h(z)| < x^{-2\mu}\varepsilon(x)\, e^{c(\eta_v-\eta_u)y}, \qquad z = x + iy, \qquad |z| \to \infty, \tag{11}$$

uniformly in any finite interval of x, where $c < 2$, $\varepsilon(x) \to 0$ for $x \to +\infty$.

Now, after the substitution of the functions (9) into formula (8), we see that for a function $h(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant a$ and satisfying condition (11), the following formula is obtained:

$$\lim_{b\to\infty}\left\{\sum_{k=m}^{n} \frac{h(z)}{\partial_z X^{\mu}_{iz}(u,v)} \frac{Q^{\mu}_{iz-1/2}(v)}{Q^{\mu}_{iz-1/2}(u)}\bigg|_{z=z_k} + \frac{i}{\pi}\mathrm{p.v.}\int_a^b dx\, f(x) + r[h(z)]\right\}$$
$$= \frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} dz\, \frac{Q^{\mu}_{-\sigma(z)iz-1/2}(v)}{Q^{\mu}_{-\sigma(z)iz-1/2}(u)} \frac{h(z)}{X^{\mu}_{iz}(u,v)}, \tag{12}$$

where the function $f(z)$ is defined by relation (9). In this formula we have introduced the notation

$$r[h(z)] = \sum_{k,\mathrm{Im}\, z_{h,k}\neq 0} \operatorname*{Res}_{z=z_{h,k}}\left[\frac{Q^{\mu}_{-\sigma(z_k)iz-1/2}(v)}{Q^{\mu}_{-\sigma(z_k)iz-1/2}(u)} \frac{h(z)}{X^{\mu}_{iz}(u,v)}\right]$$
$$+ \frac{1}{2}\sum_{k,\mathrm{Im}\, z_{h,k}=0} \operatorname*{Res}_{z=z_{h,k}}\left[\frac{h(z)}{X^{\mu}_{iz}(u,v)} \sum_{l=\pm} \frac{Q^{\mu}_{liz-1/2}(v)}{Q^{\mu}_{liz-1/2}(u)}\right], \tag{13}$$

with $z_{h,k}$ being the positions of the poles for the function $h(z)$. On the left-hand side of (12), one has $z_{m-1} < a < z_m$, $z_n < b < z_{n+1}$ and in (13) the summation goes over the poles $z_{h,k}$ in the strip $a < \mathrm{Re}\, z < b$. Note that one has the relations

$$\frac{Q^{\mu}_{iz-1/2}(v)}{Q^{\mu}_{iz-1/2}(u)} = \frac{P^{\mu}_{iz-1/2}(v)}{P^{\mu}_{iz-1/2}(u)} = \frac{P^{-\mu}_{iz-1/2}(v)}{P^{-\mu}_{iz-1/2}(u)}, \qquad z = z_k, \tag{14}$$

and in the summation of the first term in figure braces of (12) we can replace the ratio of the associated Legendre functions of the second kind by the ratio of the functions of the first kind.

A useful form of the summation formula (12) is obtained in the limit $a \to 0$. In this limit, we see that for a function $h(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant 0$ and satisfying the condition (11) the following formula holds:

$$\sum_{k=1}^{\infty} \frac{h(z)}{\partial_z X^{\mu}_{iz}(u,v)} \frac{Q^{\mu}_{iz-1/2}(v)}{Q^{\mu}_{iz-1/2}(u)}\bigg|_{z=z_k} = \frac{\pi e^{2i\mu\pi}}{4}\mathrm{p.v.}\int_0^{\infty} dx\, \frac{\Gamma(ix+\mu+1/2)\sinh(x\pi)}{\Gamma(ix-\mu+1/2)\cos[(ix-\mu)\pi]}$$
$$\times \frac{h(x)}{Q^{\mu}_{ix-1/2}(u)Q^{\mu}_{-ix-1/2}(u)} - r[h(z)]$$
$$- \frac{1}{2\pi}\int_0^{\infty} dx\, \frac{Q^{\mu}_{x-1/2}(v)}{Q^{\mu}_{x-1/2}(u)} \frac{h(x\, e^{\pi i/2}) + h(x\, e^{-\pi i/2})}{X^{\mu}_{x}(u,v)}. \tag{15}$$

By using the asymptotic formulas for the associated Legendre functions and the relation (7), for the corresponding asymptotic behavior of the function $X_x^{\mu}(u, v)$ for large values $x \gg 1$, one finds

$$X_x^{\mu}(u, v) \approx \frac{\mathrm{e}^{(\eta_v - \eta_u)x}}{\pi x \sqrt{\sinh \eta_u \sinh \eta_v}}. \tag{16}$$

From this asymptotic formula it follows that, under condition (11) for the function $h(z)$, the second integral on the right-hand side of formula (15) exponentially converges in the upper limit.

If the function $h(z)$ has poles on the positive real axis, it is assumed that the first integral on the right-hand side converges in the sense of the principal value. From the derivation of (15) it follows that this formula may be extended to the case of some functions $h(z)$ having branch points on the imaginary axis, for example, having the form $h(z) = h_1(z)/(z^2 + c^2)^{1/2}$, where $h_1(z)$ is a meromorphic function. This type of function appears in the physical example discussed in section 4. Special cases of formula (15) are considered in the next section.

Another generalization of formula (15) can be given for a class of functions $h(z)$ having purely imaginary poles at the points $z = \pm \mathrm{i}y_k$, $y_k > 0$, $k = 1, 2, \ldots$, and at the origin $z = y_0 = 0$. We assume that the function $h(z)$ satisfies the condition

$$h(z) = -h(z\mathrm{e}^{-\pi \mathrm{i}}) + o((z - \sigma_k)^{-1}), \qquad z \to \sigma_k, \qquad \sigma_k = 0, \qquad \mathrm{i}y_k. \tag{17}$$

In the way similar to that used in [5], it can be seen that formula (15) is extended for this class of functions adding to the right-hand side the sum of residues

$$-\sum_{\sigma_k = 0, \mathrm{i}y_k} (1 - \delta_{0\sigma_k}/2) \operatorname*{Res}_{z=\sigma_k} \left[\frac{Q^{\mu}_{-\mathrm{i}z-1/2}(v)}{Q^{\mu}_{-\mathrm{i}z-1/2}(u)} \frac{h(z)}{X^{\mu}_{\mathrm{i}z}(u, v)} \right], \tag{18}$$

and taking the principal value of the second integral on the right-hand side of (15). The latter exists due to condition (17).

3. Special cases

First we consider the case $\mu = 1/2$. For the corresponding associated Legendre functions one has

$$P^{-1/2}_{z-1/2}(\cosh \eta) = \sqrt{\frac{2}{\pi}} \frac{\sinh(z\eta)}{z\sqrt{\sinh \eta}}, \qquad P^{1/2}_{z-1/2}(\cosh \eta) = \sqrt{\frac{2}{\pi}} \frac{\cosh(z\eta)}{\sqrt{\sinh \eta}}. \tag{19}$$

By making use of these formulas, we find

$$X^{1/2}_{\mathrm{i}z}(u, v) = \frac{2}{\pi} \frac{\sin[z(\eta_v - \eta_u)]}{z\sqrt{\sinh \eta_u \sinh \eta_v}}. \tag{20}$$

Hence, in this case for the zeros z_k one has $z_k = \pi k/(\eta_v - \eta_u)$. Introducing a new function $F(z)$ in accordance with the relation $zh(z) = F(z(\eta_v - \eta_u)/\pi)$, from formula (15) we obtain the Abel–Plana summation formula in its standard form.

Now let us show that from formula (15), as a special case, a summation formula is obtained for the series over zeros of the combination of cylinder functions. First of all, by making use of formulas

$$\lim_{s \to +\infty} (sz)^{\pm\mu} P^{\mp\mu}_{\mathrm{i}sz-1/2}(\cosh(\lambda/s)) = J_{\pm\mu}(\lambda z), \tag{21}$$

with $J_{\mu}(\eta)$ being the Bessel function of the first kind, we can see that the following relation holds:

$$\lim_{s \to +\infty} X^{\mu}_{\mathrm{i}sz}(\cosh(\lambda_u/s), \cosh(\lambda_v/s)) = C_{\mu}(\lambda_u z, \lambda_v z), \tag{22}$$

where

$$C_\mu(\lambda_u z, \lambda_v z) = J_\mu(\lambda_u z)Y_\mu(\lambda_v z) - Y_\mu(\lambda_u z)J_\mu(\lambda_v z). \tag{23}$$

Note that, instead of the function $J_{-\mu}(z)$ we have introduced the Neumann function $Y_\mu(z)$. Hence, in the limit $s \to \infty$ from (15) we obtain the summation formula for the series over zeros $z = \lambda_{\mu,k}$, $k = 1, 2, \ldots$, of the function $C_\mu(\lambda_u z, \lambda_v z)$. For this, first we rewrite formula (15) making the replacements $z \to sz$, $x \to sx$, in both sides of this formula including the terms in $r[h(z)]$, and we take $u = \cosh(\lambda_u/s)$, $v = \cosh(\lambda_v/s)$. Introducing a new function $F(z) = h(sz)$, in the limit $s \to +\infty$ we find the formula

$$\sum_{k=1}^{\infty} \frac{F(z)}{\partial_z C_\mu(\lambda_u z, \lambda_v z)} \frac{J_\mu(\lambda_v z)}{J_\mu(\lambda_u z)}\bigg|_{z=\lambda_{\mu,k}} = \frac{1}{\pi}\text{p.v.}\int_0^\infty \mathrm{d}x\, \frac{F(x)}{J_\mu^2(\lambda_u x) + Y_\mu^2(\lambda_u x)}$$
$$-r_C[F(z)] - \frac{1}{4}\int_0^\infty \mathrm{d}x\, \frac{K_\mu(\lambda_v x)}{K_\mu(\lambda_u x)} \frac{F(x\mathrm{e}^{\pi\mathrm{i}/2}) + F(x\mathrm{e}^{-\pi\mathrm{i}/2})}{K_\mu(\lambda_u x)I_\mu(\lambda_v x) - I_\mu(\lambda_u x)K_\mu(\lambda_v x)}, \tag{24}$$

where $I_\mu(x)$ and $K_\mu(x)$ are the modified Bessel functions and

$$r_C[F(z)] = \pi\sum_k \underset{\operatorname{Im} z_{F,k}=0}{\operatorname{Res}}\left[\frac{J_\mu(\lambda_u z)J_\mu(\lambda_v z) + Y_\mu(\lambda_u z)Y_\mu(\lambda_v z)}{J_\mu^2(\lambda_u x) + Y_\mu^2(\lambda_u x)} \frac{F(z)}{C_\mu(\lambda_u z, \lambda_v z)}\right]$$
$$+\pi\sum_{l=1,2}\sum_k \underset{(-1)^l \operatorname{Im} z_{F,k}<0}{\operatorname{Res}}\left[\frac{H_\mu^{(l)}(\lambda_v z)}{H_\mu^{(l)}(\lambda_u z)} \frac{F(z)}{C_\mu(\lambda_u z, \lambda_v z)}\right]. \tag{25}$$

In deriving (24) we have also used the formulas

$$\lim_{\nu\to+\infty} \nu^{-\mu} Q_{i\nu-1/2}^{\mu}(\cosh(\eta/\nu)) = -\frac{\pi\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}\mu\pi} H_\mu^{(2)}(\eta),$$
$$\lim_{\nu\to\infty} \nu^{\pm\mu} P_\nu^{\mp\mu}(\cosh(x/\nu)) = I_{\pm\mu}(x), \tag{26}$$
$$\lim_{\nu\to\infty} \nu^{-\mu} Q_\nu^{\mu}(\cosh(x/\nu)) = \mathrm{e}^{\mathrm{i}\mu\pi} K_\mu(x),$$

and the relation

$$\frac{H_\mu^{(2)}(\lambda_v z)}{H_\mu^{(2)}(\lambda_u z)} = \frac{J_\mu(\lambda_v z)}{J_\mu(\lambda_u z)}, \qquad z = \lambda_{\mu,k}. \tag{27}$$

Note that from (26) it follows that

$$\lim_{s\to+\infty} X_{sx}^{\mu}(\cosh(\lambda_u/s), \cosh(\lambda_v/s)) = \frac{2}{\pi}[K_\mu(\lambda_u x)I_\mu(\lambda_v x) - I_\mu(\lambda_u x)K_\mu(\lambda_v x)]. \tag{28}$$

Formula (24) is a special case of the result derived in [6] (see also [8]). Physical applications of this formula are given in [10, 11].

4. Vacuum polarization by concentric spherical boundaries in a constant curvature space

4.1. Wightman function

In this section we give a physical application of the summation formula (15). Consider a scalar field $\varphi(x)$ on the background of the space with constant negative curvature described by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2[\mathrm{d}r^2 + \sinh^2 r(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)], \tag{29}$$

where a is a constant. The field equation has the form

$$\left(\nabla_l \nabla^l + M^2 + \xi R\right)\varphi(x) = 0, \tag{30}$$

where M is the mass of the field quanta, ξ is the curvature coupling parameter and for the Ricci scalar one has $R = -6a^{-2}$. We will assume that the field operator satisfies the Dirichlet boundary conditions on two concentric spherical shells with radii $r = r_1$ and $r = r_2$, $r_1 < r_2$,

$$\varphi(x)|_{r=r_{1,2}} = 0. \tag{31}$$

The boundary conditions modify the spectrum of the zero-point fluctuations and, as a result of this modification, the physical properties of the vacuum are changed. Among the most important characteristics of these properties are the expectation values of quantities bilinear in the field operator such as the field squared and the energy–momentum tensor. These expectation values are obtained from two-point functions in the coincidence limit of the arguments. As a two-point function here we will consider the positive frequency Wightman function. Other two-point functions are evaluated in a similar way. Expanding the field operator over the complete set $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ of classical solutions to the field equation satisfying the boundary conditions (31), the Wightman function is presented in the form of the following mode-sum:

$$W(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_\alpha \varphi_\alpha(x)\varphi_\alpha^*(x'), \tag{32}$$

where $|0\rangle$ is the amplitude of the vacuum state and α is a set of quantum numbers specifying the solution.

In accordance with the spherical symmetry of the problem under consideration, the eigenfunctions for the scalar field can be presented in the factorized form

$$\varphi_\alpha(x) = Z(r)Y_{lm}(\theta, \phi)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{33}$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics with $l = 0, 1, 2, \ldots, -l \leqslant m \leqslant l$. The equation for the radial function is obtained from the field equation (30) and has the form

$$\frac{1}{\sinh^2 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\sinh^2 r\frac{\mathrm{d}Z}{\mathrm{d}r}\right) + \left[(\omega^2 - M^2)a^2 + 6\xi - \frac{l(l+1)}{\sinh^2 r}\right]Z = 0. \tag{34}$$

In the region between the spherical shells the solution of equation (34) is expressed in terms of the associated Legendre function as

$$Z(r) = \frac{c_1 P_{\mathrm{i}z-1/2}^{-l-1/2}(u) + c_2 P_{\mathrm{i}z-1/2}^{l+1/2}(u)}{\sqrt{\sinh r}},$$

with integration constants c_1 and c_2 and the notations

$$z^2 = (\omega^2 - M^2)a^2 + 6\xi - 1, \qquad u = \cosh r. \tag{35}$$

From the boundary condition on the inner sphere we find

$$\frac{c_2}{c_1} = -\frac{P_{\mathrm{i}z-1/2}^{-l-1/2}(u_1)}{P_{\mathrm{i}z-1/2}^{l+1/2}(u_1)}, \qquad u_i \equiv \cosh r_i, \qquad i = 1, 2, \tag{36}$$

and hence,

$$Z(r) = C_\alpha \frac{X_{\mathrm{i}z}^{l+1/2}(u_1, u)}{\sqrt{\sinh r}}, \tag{37}$$

where C_α is the normalization constant and the function $X_{\mathrm{i}z}^{l+1/2}(u_1, u)$ is defined by (1). From the boundary condition on the outer sphere we see that the eigenvalues for z are solutions of the equation

$$X_{\mathrm{i}z}^{l+1/2}(u_1, u_2) = 0. \tag{38}$$

As a result, the eigenfunctions have the form

$$\varphi_\alpha(x) = \frac{C_\alpha}{\sqrt{\sinh r}} X_{\mathrm{i}z}^{l+1/2}(u_1, u) Y_{lm}(\theta, \phi)\, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{39}$$

and hence, $z = z_k$, $k = 1, 2, \ldots$, in the notations of section 2. The corresponding eigenfrequencies are related to these zeros by the formula

$$\omega_k^2 = \omega^2(z_k) = \left(z_k^2 + 1 - 6\xi\right)/a^2 + M^2. \tag{40}$$

Hence, the set α of the quantum numbers is specified to $\alpha = (l, m, k)$.

The coefficient C_α in (39) is determined from the orthonormalization condition for the eigenfunctions:

$$\int \mathrm{d}^3x\, \sqrt{|g|}\varphi_\alpha(x)\varphi_{\alpha'}^*(x) = \frac{\delta_{\alpha\alpha'}}{2\omega}, \tag{41}$$

where the integration goes over the region between the spherical shells. Making use of the integration formula given in the appendix and the boundary conditions, for this coefficient we find

$$C_\alpha^{-2} = a^3 \frac{\omega(z)}{z}\left(u_2^2 - 1\right)\left[\partial_z X_{\mathrm{i}z}^{l+1/2}(u_1, u_2)\right]\partial_u X_{\mathrm{i}z}^{l+1/2}(u_1, u), \tag{42}$$

with $z = z_k$, $u = u_2$. By using the Wronskian relation for the associated Legendre functions,

$$W\left\{P_{\mathrm{i}\nu-1/2}^{\mu}(u), Q_{\mathrm{i}\nu-1/2}^{\mu}(u)\right\} = \frac{\mathrm{e}^{\mathrm{i}\mu\pi}\Gamma(\mathrm{i}\nu + \mu + 1/2)}{(1 - u^2)\Gamma(\mathrm{i}\nu - \mu + 1/2)}, \tag{43}$$

it can be seen that

$$\left[\partial_u X_{\mathrm{i}z_k}^{l+1/2}(u_1, u)\right]_{u=u_2} = \frac{2}{\pi}\frac{1}{u_2^2 - 1}\frac{P_{\mathrm{i}z_k-1/2}^{l+1/2}(u_1)}{P_{\mathrm{i}z_k-1/2}^{l+1/2}(u_2)}. \tag{44}$$

Upon substituting this into (42), the normalization coefficient is written in the equivalent form

$$C_\alpha^{-2} = a^3 \frac{2\omega(z)}{\pi z}\partial_z X_{\mathrm{i}z}^{l+1/2}(u_1, u_2)\frac{P_{\mathrm{i}z-1/2}^{l+1/2}(u_1)}{P_{\mathrm{i}z-1/2}^{l+1/2}(u_2)}\bigg|_{z=z_k}. \tag{45}$$

Note that the ratio of the gamma functions in this formula can also be presented in the form

$$\frac{\Gamma(\mathrm{i}z_k + l + 1)}{\Gamma(\mathrm{i}z_k - l)} = \frac{1}{\pi}\cos[\pi(\mathrm{i}z_k - l - 1/2)]|\Gamma(\mathrm{i}z_k + l + 1)|^2. \tag{46}$$

Substituting the eigenfunctions into the mode-sum formula (32) and using the addition theorem for the spherical harmonics, for the Wightman function one finds

$$W(x, x') = \frac{1}{8a^3}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r \sinh r'}} \times \sum_{k=1}^{\infty} z\frac{X_{\mathrm{i}z}^{l+1/2}(u_1, u)X_{\mathrm{i}z}^{l+1/2}(u_1, u')}{\partial_z X_{\mathrm{i}z}^{l+1/2}(u_1, u_2)}\frac{P_{\mathrm{i}z-1/2}^{l+1/2}(u_2)}{P_{\mathrm{i}z-1/2}^{l+1/2}(u_1)}\frac{\mathrm{e}^{-\mathrm{i}\omega(z)\Delta t}}{\omega(z)}\bigg|_{z=z_k}, \tag{47}$$

where $\Delta t = t - t'$ and $u' = \cosh r'$. In (47), $P_l(\cos\gamma)$ is the Legendre polynomial and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{48}$$

As the expressions for the zeros z_k are not explicitly known, formula (47) for the Wightman function is not convenient. In addition, the terms in the sum are highly oscillatory for large values of quantum numbers.

For the further evaluation of the Wightman function we apply to the series over k the summation formula (15) with $u = u_1$ and $v = u_2$, taking in this formula

$$h(z) = zX_{\mathrm{i}z}^{l+1/2}(u_1, u)X_{\mathrm{i}z}^{l+1/2}(u_1, u')\frac{\mathrm{e}^{-\mathrm{i}\omega(z)\Delta t}}{\omega(z)}, \tag{49}$$

where the function $\omega(z)$ is defined by (40). The function (49) has no poles in the right half-plane and, hence, $r[h(z)] = 0$. The corresponding conditions are satisfied if $r+r'+\Delta t/a < 2r_2$. In particular, this is the case in the coincidence limit $t = t'$ for the region under consideration. For the function (49) the part of the integral on the right-hand side of formula (15) over the region $(0, x_M)$ vanishes, and for the Wightman function one finds

$$W(x, x') = W_1(x, x') - \frac{1}{8\pi a^2}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r\sinh r'}}\int_{x_M}^{\infty}\mathrm{d}x\, x$$
$$\times\frac{Q_{x-1/2}^{l+1/2}(u_2)}{Q_{x-1/2}^{l+1/2}(u_1)}\frac{X_x^{l+1/2}(u_1, u)X_x^{l+1/2}(u_1, u')}{X_x^{l+1/2}(u_1, u_2)}\frac{\cosh\left(\sqrt{x^2-x_M^2}\Delta t/a\right)}{\sqrt{x^2-x_M^2}}, \tag{50}$$

where we have defined

$$x_M = \sqrt{M^2a^2 + 1 - 6\xi}. \tag{51}$$

In formula (50), the first term on the right-hand side is given by

$$W_1(x, x') = -\frac{1}{32a^3}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r\sinh r'}}\int_0^{\infty}\mathrm{d}x\, x\sinh(x\pi)$$
$$\times|\Gamma(\mathrm{i}x+l+1)|^2\frac{X_{\mathrm{i}x}^{l+1/2}(u_1, u)X_{\mathrm{i}x}^{l+1/2}(u_1, u')}{Q_{\mathrm{i}x-1/2}^{l+1/2}(u_1)Q_{-\mathrm{i}x-1/2}^{l+1/2}(u_1)}\frac{\mathrm{e}^{-\mathrm{i}\omega(x)\Delta t}}{\omega(x)}. \tag{52}$$

This function does not depend on the outer sphere radius whereas the second term in (50) vanishes in the limit $r_2 \to \infty$. Hence, the two-point function given by (52) is the Wightman function for a scalar field in background spacetime described by the line element (29) outside a single sphere with radius r_1 on which the field obeys the Dirichlet boundary condition. This can also be seen by the direct evaluation using the corresponding eigenfunctions. Thus, we can interpret the second term on the right-hand side of (50) as the part in the Wightman function induced by the presence of the outer sphere.

An alternative form for the function (52) is obtained by making use of the identity

$$\frac{X_{\mathrm{i}x}^{l+1/2}(u_1, u)X_{\mathrm{i}x}^{l+1/2}(u_1, u')}{Q_{\mathrm{i}x-1/2}^{l+1/2}(u_1)Q_{-\mathrm{i}x-1/2}^{l+1/2}(u_1)} = -\frac{4}{\pi^2}P_{\mathrm{i}x-1/2}^{-l-1/2}(u)P_{\mathrm{i}x-1/2}^{-l-1/2}(u')$$
$$-\frac{4\mathrm{i}}{\pi^3}P_{\mathrm{i}x-1/2}^{-l-1/2}(u_1)\sum_{\sigma=\pm1}\frac{Q_{\sigma\mathrm{i}x-1/2}^{-l-1/2}(u)Q_{\sigma\mathrm{i}x-1/2}^{-l-1/2}(u')}{Q_{\sigma\mathrm{i}x-1/2}^{-l-1/2}(u_1)}. \tag{53}$$

Substituting (53) into (52), we can see that the part with the first term on the right-hand side of formula (53),

$$W_0(x, x') = \frac{1}{8\pi^2a^3}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r\sinh r'}}\int_0^{\infty}\mathrm{d}x\, x\sinh(\pi x)$$
$$\times|\Gamma(\mathrm{i}x+l+1)|^2P_{\mathrm{i}x-1/2}^{-l-1/2}(\cosh r)P_{\mathrm{i}x-1/2}^{-l-1/2}(\cosh r')\frac{\mathrm{e}^{-\mathrm{i}\omega(x)\Delta t}}{\omega(x)}, \tag{54}$$

is the Wightman function for a scalar field on background of the constant curvature space without boundaries (see [5]). In the part with the second term on the right-hand side of formula (53) we rotate the contour of integration over x by the angle $\pi/2$ for the term with $\sigma = -1$ and by the angle $-\pi/2$ for the term with $\sigma = 1$. As a result, the exterior Wightman function for a single spherical boundary is presented in the decomposed form

$$W_1(x, x') = W_0(x, x') - \frac{\mathrm{i}}{4\pi^2 a^2} \sum_{l=0}^{\infty} (-1)^l \frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r \sinh r'}} \int_{x_M}^{\infty} \mathrm{d}x\, x \frac{\Gamma(x+l+1)}{\Gamma(x-l)} \times \frac{P_{x-1/2}^{-l-1/2}(u_1)}{Q_{x-1/2}^{-l-1/2}(u_1)} Q_{x-1/2}^{-l-1/2}(\cosh r) Q_{x-1/2}^{-l-1/2}(\cosh r') \frac{\cosh\left(\sqrt{x^2 - x_M^2}\Delta t/a\right)}{\sqrt{x^2 - x_M^2}}, \tag{55}$$

where the second term on the right-hand side is induced by the spherical boundary. The Wightman function for the region inside a single spherical shell is investigated in [5]. The corresponding expression is obtained from (55) by the replacements $P_{x-1/2}^{-l-1/2} \rightleftarrows Q_{x-1/2}^{-l-1/2}$ in the second term on the right of this formula.

Taking the limit $a \to \infty$ with fixed $ar = R$, from the formulas given above we obtain the corresponding results for spherical boundaries in the Minkowski spacetime with radii $R_1 = ar_1$ and $R_2 = ar_2$. Note that in this limit one has $x_M = aM$ and the result does not depend on the curvature coupling parameter. Introducing a new integration variable $y = x/a$ and using the asymptotic formula for the gamma function for large values of the argument, from (50) we find

$$W^{(\mathrm{M})}(x, x') = W_1^{(\mathrm{M})}(x, x') - \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\gamma)}{4\pi^2\sqrt{RR'}} \int_M^{\infty} \mathrm{d}y\, y \frac{\cosh(\sqrt{y^2 - M^2}\Delta t)}{\sqrt{y^2 - M^2}} \times \frac{K_{l+1/2}(R_2 y)}{K_{l+1/2}(R_1 y)} \frac{G_{l+1/2}(R_1 y, Ry) G_{l+1/2}(R_1 y, R'y)}{G_{l+1/2}(R_1 y, R_2 y)}, \tag{56}$$

where we have introduced the notation $G_\nu(x, y) = K_\nu(x)I_\nu(y) - K_\nu(y)I_\nu(x)$. The first term on the right-hand side of formula (56) is the Wightman function in the region outside a single spherical boundary with radius R_1 in the Minkowski bulk. This function is given by the expression

$$W_1^{(\mathrm{M})}(x, x') = W_0^{(\mathrm{M})}(x, x') - \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\gamma)}{4\pi^2\sqrt{RR'}} \int_M^{\infty} \mathrm{d}y\, y \times K_{l+1/2}(Ry) K_{l+1/2}(R'y) \frac{I_{l+1/2}(R_1 y)}{K_{l+1/2}(R_1 y)} \frac{\cosh(\sqrt{y^2 - M^2}\Delta t)}{\sqrt{y^2 - M^2}}. \tag{57}$$

Expressions (56) and (57) are special cases of the general formulas given in [10] for a scalar field with Robin boundary conditions in arbitrary number of spatial dimensions.

4.2. *Vacuum expectation value of the field squared*

The vacuum expectation value of the field squared is obtained from the Wightman function taking the coincidence limit of the arguments. This limit is divergent and some renormalization procedure is necessary. Here the important point is that for points outside the spherical shells the local geometry is the same as for the case of without boundaries and, hence, the structure of the divergences is the same as well. This is also directly seen from formulas (50) and (55), where the second terms on the right-hand sides are finite in the coincidence limit. Since in these formulas we have already explicitly subtracted the boundary-free part, the renormalization is reduced to that for the geometry without boundaries. In this way for the renormalized vacuum expectation value of the field squared one has

$$\langle\varphi^2\rangle_{\rm ren} = \langle\varphi^2\rangle_{1,\rm ren} - \frac{1}{8\pi a^2}\sum_{l=0}^{\infty}\frac{2l+1}{\sinh r}\int_{x_M}^{\infty}{\rm d}x \times \frac{x}{\sqrt{x^2-x_M^2}}\frac{Q_{x-1/2}^{l+1/2}(u_2)}{Q_{x-1/2}^{l+1/2}(u_1)}\frac{\left[X_x^{l+1/2}(u_1,u)\right]^2}{X_x^{l+1/2}(u_1,u_2)}, \tag{58}$$

where the first term on the right-hand side is the corresponding quantity outside a spherical boundary with radius r_1 in the constant negative curvature space without boundaries and the second one is induced by the presence of the second spherical shell with the radius r_2. Note that the latter vanishes on the interior sphere. For the first term one has

$$\langle\varphi^2\rangle_{1,\rm ren} = \langle\varphi^2\rangle_{0,\rm ren} - \sum_{l=0}^{\infty}\frac{{\rm e}^{{\rm i}(l+1/2)\pi}}{4\pi^2a^2}\frac{(2l+1)}{\sinh r}\int_{x_M}^{\infty}{\rm d}x\, x \times \frac{\Gamma(x+l+1)}{\Gamma(x-l)}\frac{P_{x-1/2}^{-l-1/2}(u_1)}{Q_{x-1/2}^{-l-1/2}(u_1)}\frac{\left[Q_{x-1/2}^{-l-1/2}(u)\right]^2}{\sqrt{x^2-x_M^2}}, \tag{59}$$

where $\langle\varphi^2\rangle_{0,\rm ren}$ is the vacuum expectation value for the field squared in the constant negative curvature space without boundaries and the second one is induced by the presence of a single spherical shell with radius r_1. Note that the corresponding formula for the vacuum expectation value inside a spherical shell (see [5]) is obtained from (59) by the replacements $P_{x-1/2}^{-l-1/2} \rightleftarrows Q_{x-1/2}^{-l-1/2}$ in the second term on the right-hand side.

Let us discuss the behavior of the vacuum expectation value of the field squared in the asymptotic regions of the parameters. First we consider the part corresponding to the geometry of a single sphere with radius r_1. In the exterior region this expectation value is given by formula (59). For points far from the sphere, $r \gg 1$, for the associated Legendre function we have

$$Q_{x-1/2}^{\mu}(\cosh r) \approx \sqrt{\pi}\,{\rm e}^{{\rm i}\mu\pi}\frac{\Gamma(x+\mu+1/2)}{\Gamma(x+1)}{\rm e}^{-(x+1/2)r}, \tag{60}$$

and the main contribution in the integral on the right-hand side of formula (59) comes from the region near the lower limit. To the leading order one finds

$$\langle\varphi^2\rangle_{1,\rm ren} \approx \langle\varphi^2\rangle_{0,\rm ren} - \frac{\sqrt{x_M/r}\,{\rm e}^{-2r(1+x_M)}}{4\sqrt{\pi}a^2\Gamma(1+x_M)}\sum_{l=0}^{\infty}(2l+1) \times \frac{\Gamma(x_M+l+1)\Gamma(x_M-l)}{{\rm e}^{{\rm i}(l+1/2)\pi}}\frac{P_{x_M-1/2}^{-l-1/2}(u_1)}{Q_{x_M-1/2}^{-l-1/2}(u_1)}, \tag{61}$$

and the expectation value of the field squared is exponentially suppressed. In particular, for a minimally coupled scalar field the suppression is stronger than in the case of the conformal coupling. For large values l the terms of the series in (61) behave like $4l^{2x_M+1}\tanh^{2l+1}(r_1/2)$. Note that the limit under consideration corresponds to large proper distances from the sphere compared to the curvature radius of the background geometry. For the geometry of spherical boundary in the Minkowski bulk the vacuum expectation value of the field squared at large distances from the sphere decays exponentially for a massive field and as a power law for a massless field. In the limit $r_1 \to 0$ with fixed r we use the asymptotic formulas for the associated Legendre functions for the arguments close to 1. The main contribution in (59) comes from the term $l = 0$ and in the leading order we obtain

$$\langle\varphi^2\rangle_{1,\mathrm{ren}} \approx \langle\varphi^2\rangle_{0,\mathrm{ren}} - \frac{x_M r_1 K_1(2x_M r)}{4\pi^2 a^2 \sinh^2 r}, \qquad r_1 \ll 1. \tag{62}$$

Now let us consider the behavior of the vacuum expectation value inside a single spherical shell in the limit $r_1 \to \infty$ when r is fixed. Using the asymptotic formula (60) with $r = r_1$ and the corresponding formula

$$P^{\mu}_{x-1/2}(\cosh r_1) \approx \frac{\pi^{-1/2}\Gamma(x)\mathrm{e}^{(x-1/2)r_1}}{\Gamma(x-\mu+1/2)}, \tag{63}$$

for the associated Legendre function of the first kind, we can see that the main contribution to the integral comes from the region near the lower limit. To the leading order we find

$$\langle\varphi^2\rangle_{1,\mathrm{ren}} \approx \langle\varphi^2\rangle_{0,\mathrm{ren}} - \frac{x_M\sqrt{x_M/r_1}\,\mathrm{e}^{-2x_M r_1}}{8\sqrt{\pi}a^2\sinh r\,\Gamma^2(1+x_M)} \times \sum_{l=0}^{\infty}(2l+1)\Gamma^2(x_M+l+1)\left[P^{-l-1/2}_{x_M-1/2}(u)\right]^2. \tag{64}$$

For large values of l, the separate terms of the series behave as $2l^{2x_M}\tanh^{2l+1}(r/2)$. In the case $x_M = 0$ and for large values r_1 we have

$$\langle\varphi^2\rangle_{1,\mathrm{ren}} \approx \langle\varphi^2\rangle_{0,\mathrm{ren}} - \frac{1}{16\pi a^2 r_1^2}\sum_{l=0}^{\infty}\frac{(2l+1)}{\sinh r}\left[l!P^{-l-1/2}_{-1/2}(u)\right]^2. \tag{65}$$

In this case we have power law decay of the sphere-induced part as a function of the physical radius of the sphere ar_1.

The asymptotic behavior of the second term on the right-hand side of (58) is investigated in the similar way. For large values r_2 when r is fixed, this term is suppressed by the factor $\mathrm{e}^{-2x_M r_2}$ for $x_M \neq 0$ and behaves as r_2^{-2} for $x_M = 0$. By using the properties of the associated Legendre functions, it may be checked that in the limit $r_1 \to 0$ for fixed values r the second term on the right-hand side of (58) coincides with the vacuum expectation value induced by a single spherical boundary with radius r_2 in the interior region.

The physical example discussed in this section demonstrates the advantages for the application of the Abel–Plana-type formulas in the evaluation of the expectation values of local physical observables in the presence of boundaries. For the summation of the corresponding mode-sums the explicit form of the eigenfrequencies is not necessary and the part corresponding to the boundary-free space is explicitly extracted. Further, the boundary-induced parts are presented in the form of integrals which rapidly converge and are finite in the coincidence limit for points away from the boundaries. In this way the renormalization procedure for local physical observables is reduced to that in quantum field theory without boundaries. Methods for the evaluation of global characteristics of vacuum, such as the total Casimir energy, in problems where the eigenmodes are given implicitly as zeros of a given function, are described in [12].

5. Conclusion

The associated Legendre functions arise in many problems of mathematical physics. By making use of the generalized Abel–Plana formula, we have derived summation formula (15) for the series over the zeros of the combination (1) of the associated Legendre functions with respect to the degree. This formula is valid for functions $h(z)$ meromorphic in the right half-plane and obeying condition (11). The summation formula may be extended to a class of functions having purely imaginary poles and satisfying the condition (17). For this, on the right-hand side of (15) we have to add the sum of residues (18) and take the principal value of the second integral on the right-hand side. Using formula (15), the difference between the sum over the zeros of the combination of the associated Legendre functions and the corresponding integral is presented in terms of an integral involving the Legendre-associated functions with real values of the degree plus residue terms. For a large class of functions $h(z)$ this integral converges exponentially fast and, in particular, is useful for numerical calculations. The Abel–Plana summation formula is obtained as a special case of formula (15) with $\mu = 1/2$ and for an analytic function $h(z)$. Applying the summation formula for the series over the zeros of the function $X_{\mathrm{i}z}^{\mu}(\cosh(\lambda_u/s), \cosh(\lambda_v/s))$ and taking the limit $s \to \infty$, we have obtained formula (24) for the summation of the series over zeros of the combination of the Bessel functions. The latter is a special case of the formula, previously derived in [6].

A physical application of the summation formula is given in section 4. For a quantum scalar field with the general curvature parameter we have evaluated the positive frequency Wightman function and the vacuum expectation value of the field squared for the geometry of concentric spherical shells in a constant negative curvature space. The Dirichlet boundary conditions on both shells are assumed. In the region between the shells the eigenfunctions have the form (39) and the corresponding eigenfrequencies are related to the zeros of the function $X_{\mathrm{i}z}^{l+1/2}(u_1, u_2)$ by formula (40). For the evaluation of the corresponding series in the mode-sum (47) for the Wightman function we apply summation formula (15) with the function $h(z)$ given by (49). As a result this function is presented in the decomposed form (50), where the first term on the right-hand side is the Wightman function for the region outside a single spherical boundary and the second one is induced by the presence of the outer sphere. By making use of the identity (53), we have presented the single shell Wightman function as a sum of two terms, formula (55). The first one is the corresponding function in the constant curvature space without boundaries and the second one is induced by the shell. For points away from the shell the latter is finite in the coincidence limit and can be directly used for the evaluation of the boundary-induced part in the vacuum expectation value of the field squared. The renormalization is necessary for the boundary-free part only and this procedure is the same as that in quantum field theory without boundaries. In the region between the spherical shells the vacuum expectation value of the field squared is presented in the form (58), where the first term on the right-hand side is the corresponding quantity outside a spherical boundary and is given by the expression (59). We also investigate the behavior of the boundary-induced part in the expectation value for the field squared in various asymptotic regions of the parameters.

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Appendix. On the zeros of the function $X_{\mathrm{i}z}^{\mu}(u, v)$

In this appendix we show that the zeros $z = z_k$ are simple and real. First, we note that the functions $P_{\mathrm{i}z-1/2}^{\pm\mu}(u)$ satisfy the Legendre equation with $\nu = \mathrm{i}z - 1/2$ and, hence, the function $X_{\mathrm{i}z}^{\mu}(u, v)$ is a solution of the Legendre equation for the same value ν with respect to both arguments. As a result, by making use of the differential equation for the associated Legendre functions it can be seen that the following integration formula takes place:

$$\int_u^v \mathrm{d}u\, X^\mu_{\nu'}(u,v)X^\mu_\nu(u,v) = (1-u^2)\frac{X^\mu_{\nu'}(u,v)\partial_u X^\mu_\nu(u,v) - X^\mu_\nu(u,v)\partial_u X^\mu_{\nu'}(u,v)}{\nu'^2-\nu^2}. \tag{A.1}$$

Taking the limit $\nu' \to \nu$ and applying l'Hôpital's rule for the right-hand side, from this formula we find

$$\int_u^v \mathrm{d}u\big[X^\mu_{\mathrm{i}z}(u,v)\big]^2 = -\frac{u^2-1}{2z}\big\{\big[\partial_z X^\mu_{\mathrm{i}z}(u,v)\big]\partial_u X^\mu_{\mathrm{i}z}(u,v) - X^\mu_{\mathrm{i}z}(u,v)\partial_z\partial_u X^\mu_{\mathrm{i}z}(u,v)\big\}. \tag{A.2}$$

By taking into account the relation $X^\mu_{-\mathrm{i}z}(u,v) = X^\mu_{\mathrm{i}z}(u,v)$, we see that for real z one has $[X^\mu_{\mathrm{i}z}(u)]^2 = |X^\mu_{\mathrm{i}z}(u)|^2$ and the integral on the left-hand side of (A.2) is positive. Now from (A.2) it follows that $[\partial_z X^\mu_{\mathrm{i}z}(u,v)]_{z=z_k} \neq 0$, and hence, the zeros z_k are simple.

Now let us show that all zeros of the function $X^\mu_{\mathrm{i}z}(u,v)$ are real. Suppose that $z=\lambda$ is a zero of $X^\mu_{\mathrm{i}z}(u,v)$ which is not real. As the function $X^\mu_{\mathrm{i}z}(u,v)$ has no pure imaginary zeros, λ is not a pure imaginary. If λ^* is the complex conjugate to λ, then it is also a zero of $X^\mu_{\mathrm{i}z}(u,v)$, because $X^\mu_{\mathrm{i}\lambda^*}(u,v) = \big[X^\mu_{\mathrm{i}\lambda}(u,v)\big]^*$. As a result, from formula (A.1) we find

$$\int_u^v \mathrm{d}v\, X^\mu_{\mathrm{i}\lambda^*}(u,v)X^\mu_{\mathrm{i}\lambda}(u,v) = 0. \tag{A.3}$$

We have obtained a contradiction, since the integrand on the left-hand side is positive. Hence, the number λ cannot exist and the function $X^\mu_{\mathrm{i}z}(u)$ has no zeros which are not real.

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